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Real borcherds superalgebras and M-theory

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Abstract: The correspondence between del Pezzo surfaces and field theory models, discussed in [1] and in [2] over the complex numbers or for split real forms, is extended to other real forms, in particular to those compatible with supersymmetry. Specifically, all theories of the Magic triangle [3] that reduce to the pure supergravities in four dimensions correspond to singular real del Pezzo surfaces and the same is true for the Magic square of $\mathcal{N} = 2$ SUGRAS [4].
A real del Pezzo surface is the invariant set under an antilinear involution of a complex one. This conjugation induces an involution of the Picard group that preserves the anticanonical class and the intersection form. The known non-split U-duality algebras are embedded into Borcherds superalgebras defined by their Cartan matrix (minus the intersection form) and fixed by the anti-involution. These data may be described by Tits-Satake bicoloured superdiagrams.
As in the split case, oxidation results from blowing down disjoint real $\mathbb{RP}_1$'s of self-intersection -1. The singular del Pezzo surfaces of interest are obtained by degenerating regular surfaces upon contraction of real curves of self-intersection -2. We use the finite classification of real simple singularities to exhibit the relevant normal surfaces. We also give a general construction of more magic triangles like a type-I split magic triangle and prove their (approximate) symmetry with respect to their diagonal, this symmetry argument was announced in our previous paper for the split case.

Keywords: M-Theory, Global Symmetries, Supergravity Models, Differential and Algebraic Geometry.

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Contents

1. Introduction

Finite dimensional Lie theory has been crucial for the construction of the standard model. In the same way, symmetry algebras will be crucial to understand the structure of M-theory. Recently, we have embedded the U-duality algebra $E_{n|n}$ for eleven-dimensional supergravity compactified on a torus T^n in a Borcherds superalgebra using a connection with the del Pezzo surfaces [2]. This huge algebra in turn could be truncated to exhibit the superalgebra of symmetries of [5].

The classification of projective algebraic curves (compact connected Riemann surfaces) leads to the genus invariant and to the list: Riemann sphere, elliptic (plane) curves and quotients of hyperbolic half-plane by cocompact discrete subgroups of $\mathrm{SL}(2,\mathbb{R})$. The classification of smooth projective algebraic surfaces has also been known for a while, but the Enriques-Kodaira theory is much richer than uniformisation theory. One distinguishes [7, 8] them first by their Kodaira dimension κ which can be $-\infty, 0, 1, 2$. The first case is that of ruled surfaces (ie birationally equivalent to the product of a curve Σ_g by a projective line). The birational class of $\mathbb{CP}^2$ is the set of rational surfaces ($g = 0$). The other ruled surfaces are obtained by a set of blowups from projective line bundles on curves. The latter are called their "minimal models" for that reason. Among rational surfaces one meets first

JHEP04(2003)060

the del Pezzo surfaces but also the Veronese surface. The case of null Kodaira dimension encompasses K3 surfaces and their quotients the Enriques surfaces as well as the abelian and the bielliptic surfaces. Non rational elliptic pencils have Kodaira dimension one, and the rest are the so-called general surfaces. For nonruled surfaces, the minimal models are unique and their classification is the same as the classification up to birational equivalence. For rational surfaces, one notes that $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ are both minimal, birationally equivalent to each other but nonisomorphic surfaces.

The Picard group or second homology of the del Pezzo surfaces defines a nice lattice connected with the root lattice of the Borcherds superalgebras. These enlarged algebras require the use of a new formulation of M-theory which introduces the fields and their duals [5]. The equations of motion for the various p-forms should then be written as a self-duality equation based on a supercoset G/K_G with G the supergroup corresponding to the Borcherds superalgebra and K_G its maximal "compact" subgroup, defined as the fixed elements under the Cartan-Chevalley-Serre involution. The other U-duality algebras appearing in the oxidation of $E_{n|n}$ cosets from 3 dimensions (for $n \leq 7$) form another ("Split Magic") triangle, SMΔ, they have also been extended to split Borcherds superalgebras and connected with (singular) normal del Pezzo surfaces in [2]. The reflection symmetry along the diagonal of SMΔ was proved there.

Now the supergravity theories that give pure supergravity in four dimension with less than $\mathcal{N} = 8$ supersymmetries form the original Magic Triangle MΔ [3], the U-duality algebras include non-split real forms of simple Lie algebras. We will show in this paper how to extend these non-split forms to non-split Borcherds superalgebras too. The conjugation defining a real form of the Borcherds superalgebra will be connected with the real structure of the corresponding real normal del Pezzo surface, defined by a complex conjugation which induces an involution of the Picard group that preserves the anticanonical class. These real forms of Borcherds superalgebras will then be characterized by a Satake superdiagram. In the first part ofi this paper, as a warm-up exercise, we give the complete generalizable proof of the symmetry of the Split Magic triangle and construct yet another symmetric triangle starting from type-I supergravity.

Then we shall review the theory of real del Pezzo surfaces. Section 4 is on real forms of Lie algebras and section 5 on the (nonsplit) Magic triangle and Magic square, there we reproduce supergravity models by adding by hand appropriate singularities on the surfaces. In the conclusion and appendix we emphasize some open questions, in particular the need to understand better supersymmetry breaking in this context as well as some peculiarity of $\mathcal{N} = 2$ pure SUGRA.

2. Symmetry of triangles

It was noticed in [6] that the "Split" Magic Triangle (SMΔ) constructed by oxidizing some split non-linear σ-models has a symmetry across the diagonal (table 1). There are 5 bifurcations which we discovered in [2], they appear in the text of that paper but unfortunately not on the table so we do it here. The bifurcations reflect the lack of uniqueness of the minimal model. For completeness we should recall that another magic triangle related to

JHEP04(2003)060

	$n=8$	$n=7$	$n=6$	$n=5$	$n=4$	$n=3$	$n=2$	$n=1$	$n=0$
$d=11$	$+$								
$d=10$	$\mathbb{R}$ or A_1	$+$							
$d=9$	$\mathbb{R}\times A_1$	$\mathbb{R}$							
$d=8$	$A_1\times A_2$	$\mathbb{R}\times A_1$ or A_2	A_1						
$d=7$	E_4	$\mathbb{R}\times A_2$	$\mathbb{R}\times A_1$	$\mathbb{R}$	$+$				
$d=6$	E_5	$A_1\times A_3$	$\mathbb{R}\times A_1^2$	$\mathbb{R}^2$ or A_1^2	$\mathbb{R}$				
$d=5$	E_6	A_5	A_2^2	$\mathbb{R}\times A_1^2$	$\mathbb{R}\times A_1$	A_1			
$d=4$	E_7	D_6	A_5	$A_1\times A_3$	$\mathbb{R}\times A_2$	$\mathbb{R}\times A_1$ or A_2	$\mathbb{R}$	$+$	
$d=3$	E_8	E_7	E_6	E_5	E_4	$A_1\times A_2$	$\mathbb{R}\times A_1$	$\mathbb{R}$ or A_1	$+$

Table 1: The split magic triangle [2].

exceptional groups extending the ideas of Freudenthal and Tits has been proposed by P. Cvitanovic [9], however it does involve non-simply laced algebras and does not fit with our more primitive constructions yet. Actually it was noted long ago by the second author that the Magic square of [4] corresponds to Tits geometries of types F_4 resp C_3, A_2, A_1. So in effect non-simply laced algebras do enter in the theory of real forms of simply laced ones.

In [2] we associated to each of these theories a del Pezzo surface. These del Pezzo's are possibly singular, but the singularity is at worst Du Val, which means that there are isolated singular points which can be resolved in a set of (-2)-curves intersecting according to an ADE Cartan matrix (in fact A type singularities suffice). Before translating the symmetry of the triangle into Lie superalgebra language, we will first explain it at the level of algebraic geometry of del Pezzo surfaces.

2.1 Del Pezzo surfaces and coset theories

Let us first say a few words on del Pezzo surfaces and how they are related to physical theories. For a detailed exposition and mathematical background, see [2].

A (complex) del Pezzo surface is an algebraic, projective, complex surface which has an ample anticanonical divisor. A complex surface (which has complex dimension two) is said to be projective if it can be embedded in some $\mathbb{CP}^n$ in such a way that, at least locally, it is defined as the zero locus of some homogeneous polynomials. The maximal exterior power of the tangent bundle is a line bundle, called the anticanonical divisor and denoted $-K$. The property of being ample is that its first Chern class is positive: it gives a positive integer when integrated on any 1-cycle.

Smooth del Pezzo surfaces are $\mathbb{CP}^2$, $\mathbb{CP}^1\times\mathbb{CP}^1$ and $\mathbb{CP}^2_r$ blown up in $r\leq 8$ points in general position. Blowing up a point means replacing it by the $\mathbb{CP}^1$ of all tangent directions.

The H_2 lattice of integral cohomology (or Picard group) of a del Pezzo surface is endowed with an intersection product. We recall that the Picard group of $\mathbb{CP}^2_r$ is generated by the projective line H of $\mathbb{CP}^2$ and the r exceptional curves E_i and the Picard group of $\mathbb{CP}^1\times\mathbb{CP}^1$ is generated by the two lines l_1 and l_2 satisfying $l_1^2=l_2^2=0$ and $l_1.l_2=1$.

One can construct Du Val singularities by contracting chains of curves of self-intersection -2. The type of the singularity depends on the intersections of the (-2)-curves considered and is described by a Dynkin diagram. We will consider here A_k singularities.

In [1] it was remarked that BPS states of M-theory compactified on orthogonal tori correspond to divisors of vanishing (virtual) genus (called rational divisors) on smooth del Pezzo's. In [2] we have shown that one can associate a Borcherds generalised Cartan matrix to these surfaces, from which we were able to recover the equations of motion of all p-forms of the theories, and relations between tensions of BPS objects. Moreover, we associated to each theory of the oxidation triangle of [6] a del Pezzo surface which gives its bosonic field content and equations of motion. The symmetry of the triangle, on the del Pezzo side of the correspondence, is the fact that each pair of symmetric del Pezzo's have the same degree zero divisors. We now give the precise proof of this fact, already sketched in [2].

2.2 Proof of the triangle symmetry

Let us first recall that for del Pezzo surfaces, the symmetry of the SMΔ follows from the fact that a del Pezzo surface of degree $n+1$ ($K^2 = n+1$ with K the canonical divisor) with exactly one singularity of type A_k has the same group of reflections as the del Pezzo of degree $k+1$ and singularity A_n. This group is the group of reflections with respect to the (-2)-divisors orthogonal to K. In fact, one can check that the sublattices of the Picard group orthogonal to K, ($K^\perp$), are identical for both cases and not only the (-2)-divisors.

Let us now show that given any del Pezzo surface, at least when it is normal, we can construct a symmetric triangle such that this surface lies on the first column of it. Let us start with some given singularities on this surface, in the following we shall focus on one additional singularity. When the starting surface is non-singular we recover the case of the previous paragraph. We may first blow up points in general position in order to get a del Pezzo surface with $K^2 = 1$. We claim that if we can blow down n non-intersecting (-1)-curves isomorphic to $\mathbb{CP}^1$ on that new surface to get a del Pezzo of degree $n+1$, we can alternatively construct another del Pezzo of degree 1 with an additional singular point A_n with the same $K^\perp$, and conversely. Applying this twice, we see that we can construct a del Pezzo surface of degree $n+1$ and one singular point A_k which has the same $K^\perp$ as a del Pezzo of degree one and two singular points A_k and A_n. The same reasoning tells us that this surface has the same $K^\perp$ (full homologies may differ) as a third one of degree $k+1$ and singularity A_n, which proves the symmetry of the triangle of split U-dualities with respect to the diagonal.

Now, let us explain why for a del Pezzo surface of degree one there is a one-to-one correspondence between blowing down k non-intersecting (-1)-curves isomorphic to $\mathbb{CP}^1$ and an A_k singularity. At the level of the Picard group, blowing down curves corresponds to keeping their orthogonal complement. Let E_i, with i running from 1 to k be the so-called exceptional (-1)-spheres. As they do not intersect, we have $E_i.E_j = 0$, and of course $E_i^2 = -1$ and $K.E_i = -1$. The vector space spanned by these divisors is also generated by the divisors $E_i - E_{i+1}$ ($1 \leq i \leq k-1$) and $-E_1$. So the orthogonal complement of these two sets of divisors is the same. This is a result at the level of divisors without assurance at this stage that the second set can be realised effectively by spheres. If we restrict ourselves to $K^\perp$ in the Picard group, we can also exchange any E_i with $-E_i - K$, and we see that inside the subspace of the Picard group perpendicular to K, the orthogonal complement of the E_i's, with $1 \leq i \leq k$ is also the orthogonal complement of $E_i - E_{i+1}$, with $1 \leq i \leq k-1$

and $-E_1 - K$ which are themselves (-2)-divisors in $K^\perp$. Moreover, one can check, keeping in mind $K^2 = 1$, that these are virtual spheres in an A_k configuration in other words whose intersection matrix is the opposite of an A_k Cartan matrix.

One can easily see that this argument can be inverted and that starting with such an A_k of (-2)-divisors in the Picard group, we can find k orthogonal (-1)-divisors giving the same orthogonal complement in $K^\perp$. Now the key point is to decide which divisors are actually curves.

In order to construct an A_k singularity from these (-2)-divisors, we have to deform the del Pezzo surface such that these become effective (-2)-spheres. Let us call p_i and p_{i+1} the points we get by blowing down E_i and E_{i+1}. For $E_i - E_{i+1}$ to be a curve, we have to take p_i and p_{i+1} infinitely close, this preserves the orthogonality of E_i and E_{i+1} as it is a continuous process. In other words, p_{i+1} must lie on E_i. If the del Pezzo surface is normal, once all singularities are blown up, we get the smooth del Pezzo surface $\mathbb{CP}^2_8$, which corresponds to blowing up $\mathbb{CP}^2$ on eight almost general points, and the anticanonical divisor can be written as $-K = 3H - E_1 - E'_2 - \ldots - E'_8$ and therefore we have $-K - E_1 = 3H - 2E_1 - E'_2 - \ldots - E'_8$. This divisor is a curve if the eight points corresponding to the exceptional divisors $E_1, E'_2, \ldots E'_8$ lie on a cubic curve of $\mathbb{CP}^2$ which has a double point that we take as p_1. Thus, we can construct an A_k chain of (-2)-curves to be blown down in order to get the del Pezzo surface we wanted.

When the surface is nonnormal, the very last point is more subtle and we do not know if such an A_k singularity can always be constructed. We will see below a case where it is possible, starting with the del Pezzo surface we associated to a truncated version of type-I or Heterotic String theory.

2.3 Superalgebras

On the superalgebra side of the dictionary, the translation is the following. The symmetry of the split triangle comes from the fact that in superalgebras corresponding to σ-models in $d = 3$, the centralizer of $n - 1$ commuting $su(1|1)$ and one $sl(k + 1)$ (also commuting with the former) has the same degree 0 part as the centralizer of k $sl(1|1)$ and one $sl(n)$, all commuting with each other. In other words, instantons of both theories are the same, and so is the U-duality algebra. In the supersymmetric case the symmetry is only approximate. One should still factorise a split $sl(n)$ while going up any column ie under oxidation this is related to the fact that this algebra is the remnant of diffeomorphisms. But the rule we must adopt now for going from left to right at fixed dimension is rather to disintegrate the R-symmetry group, again starting from its affine root as in oxidation. However one is now factorising $su(n)$'s.

2.4 Split Triangle constructed from type-I

In [2] we showed that a del Pezzo surface can be associated to type-I or Heterotic theories when one forgets the gauge sector. This surface is not normal, which means that there is a singularity of (co)dimension 1. We show here that a symmetric triangle of del Pezzo surfaces can be constructed from it nevertheless, corresponding to σ-models oxidations (table 2).

JHEP04(2003)060

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$d=10$	$\mathbb{R}$							
$d=9$	$\mathbb{R}^2$							
$d=8$	$\mathbb{R}\times A_1^2$	$\mathbb{R}\times A_1$						
$d=7$	$\mathbb{R}\times A_3$	$\mathbb{R}^2\times A_1$	$\mathbb{R}^2$	$\mathbb{R}$				
$d=6$	$\mathbb{R}\times D_4$ or D_5	$\mathbb{R}\times A_1^3$ or $A_1\times A_3$	$\mathbb{R}\times A_1^2$	$\mathbb{R}^2$ or A_1^2	$\mathbb{R}$			
$d=5$	$\mathbb{R}\times D_5$	$\mathbb{R}\times A_1\times A_3$	$\mathbb{R}^2\times A_1^2$	$\mathbb{R}\times A_1^2$	$\mathbb{R}^2$			
$d=4$	$A_1\times D_6$	D_6 or $A_1^2\times D_4$	$\mathbb{R}\times A_1\times A_3$	$A_1\times A_3$ or $\mathbb{R}\times A_1^3$	$\mathbb{R}^2\times A_1$	$\mathbb{R}\times A_1$		
$d=3$	D_8	$A_1\times D_6$	$\mathbb{R}\times D_5$	$\mathbb{R}\times D_4$ or D_5	$\mathbb{R}\times A_3$	$\mathbb{R}\times A_1^2$	$\mathbb{R}^2$	$\mathbb{R}$

Table 2: Split Magic type-I-Heterotic Triangle

The starting surface can be obtained in the following manner. We start from $\mathbb{CP}^2$ and blow up one point, p_1, on it. Then we blow up a second point, p_2, lying on the corresponding exceptional divisor E_1 which gives another exceptional divisor E_2. Then there is a (-2)-curve $E_1 - E_2$, which is the transform of E_1 by the second blow-up, and a third (-1)-curve $H - E_1 - E_2$ (where H is the line of $\mathbb{CP}^2$) that we blow down. This surface is a $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ and its Picard group is generated by the section $E_1 - E_2$ and the fiber $H - E_1$. Now we blow up the intersecting point p_3 of $E_1 - E_2$ with one of the fibers, which gives an exceptional divisor E_3. The (-2)-section becomes a (-3)-curve $E_1 - E_2 - E_3$. After blowing down the (-1)-curve $H - E_1 - E_3$, we get another $\mathbb{CP}^1$-bundle over $\mathbb{CP}^1$ which still has fibers $A = H - E_1$ but which has now a (-3)-section $E_1 - E_2 - E_3$. We repeat this operation once more and get a bundle with the same fiber but with a (-4)-section $B = E_1 - E_2 - E_3 - E_4$.

This (-4)-curve B which is a conic can be projected to a double line, and we get the surface described in [2] with $-K = B + 6A$, which corresponds to the type-I theory (without gauge sector) in 10 dimensions. We get the surfaces corresponding to its toroidal compactifications by blowing up points in general positions, as in the regular case. The corresponding exceptional curves are noted E'_i. To construct a triangle on the right of this column (table 1), we have seen above that the divisor $-K' - E'_1$ must be an irreducible curve, where $-K' = B + 6A - E'_1 - E'_2 - \ldots - E'_7$ is the anticanonical divisor of the $d = 3$ surface, which is obtained by blowing up seven points (p'_1 to p'_7) on the $d = 10$ one. As we have $-K' - E'_1 = 6H - 5E_1 - E_2 - E_3 - E_4 - 2E'_1 - E'_2 - \ldots - E'_7$, we see that this is a curve if we take all the blown-up points on a sextic such that p_1 is a quintuple point and p'_1 a double point, which is possible. Indeed such a curve is defined by an homogeneous polynomial of degree 6 in 3 variables, which has 28 coefficients, which leaves 27 free coefficients after modding out by the action of $\mathbb{C}^*$, and this is larger than 16, the number of fixed points.

We observe that the del Pezzo surface ($k = 1$, $d = 6$) with Weyl group D_5 corresponds to type-IIB compactified on $T^4/\mathbb{Z}_2$ without the twisted sector. We can obtain the other surface in the same slot (with Weyl group D_4) by applying a birational transformation which corresponds to a blow up and a blow down of (-1)-curves. On the supergravity side, it corresponds to a T-duality. The field contents of the various field theories obtained includes a metric, a dilaton, a two form and some axions corresponding to the U-duality algebra.

3. Real del Pezzo surfaces

A *real algebraic variety* $X_{\mathbb{R}}$ is defined as a complex algebraic variety with an antiholomorphic involution σ_X, called the *real structure* [10, 11]. This real structure defines an involution of the Picard group $Pic(X_{\mathbb{R}})$ in the following way: if D is a Cartier divisor, it can be represented by meromorphic functions f_i on an open cover U_i of $X_{\mathbb{R}}$ (with $\frac{f_i}{f_j}$ holomorphic and non-vanishing on $U_i \cap U_j$), then $\sigma(D)$ is given by $(U_i, f_i^{\sigma_X} := j \circ f_i \circ \sigma_X)$ where $j : \mathbb{C} \to \mathbb{C}$ is the complex conjugation. The intersection of two divisors D_1 and D_2 defined as $D_1.D_2 := \int_{X_{\mathbb{R}}} c_1(D_1) \cup c_1(D_2))$ (with c_1 the first Chern class) is then preserved by the real structure: $\sigma_X(D_1).\sigma_X(D_2) = D_1.D_2$. Moreover, it can be proved that the anticanonical class K_X is invariant under σ_X.

We recall that if we blow up a point P on a complex surface $X_{\mathbb{C}}$, we obtain a surface $Y_{\mathbb{C}}$ and get a projection morphism $\pi : Y_{\mathbb{C}} \to X_{\mathbb{C}}$ such that $E = \pi^{-1}(P)$ is an exceptional curve of the first kind (that is $E \simeq \mathbb{CP}^1$ and $E.E = -1$) and $Y_{\mathbb{C}} \setminus \pi^{-1}(P) \simeq X_{\mathbb{C}} \setminus P$. The monoidal transformation π is defined as follows: let (z_1, z_2) be local coordinates at P defined in an open set U, then $V = \pi^{-1}(U)$ has equation $l_1 z_2 = l_2 z_1$ in $U \times \mathbb{CP}^1$, where (l_1, l_2) are homogenous coordinates of $\mathbb{CP}^1$. If we now blow up a point P on a real surface $X_{\mathbb{R}}$ with involution σ_X, we obtain a real surface $Y_{\mathbb{R}}$ with a natural real structure σ_Y defined by $\sigma_Y(z_1, z_2, l_1, l_2) = (\sigma_X(z_1), \sigma_X(z_2), \bar{l}_1, \bar{l}_2)$ on V where $^-$ denotes complex conjugation in $\mathbb{CP}^1$. This involution coincides with the one induced on $Y_{\mathbb{R}} \setminus \pi^{-1}(P)$ by identification with $X_{\mathbb{R}} \setminus P$. Then, if P is a real point, we obtain an exceptional curve of the first kind E satisfying $\sigma_Y(E) = E$ and for complex conjugate points P_1 and P_2, we have $\sigma_Y(E_1) = E_2$ and $E_1.E_2 = 0$ as P_1 and P_2 are distinct points. Conversely, if E is such a real (resp. a pair of complex conjugate) exceptional curve(s), $Y_{\mathbb{R}}$ can be blown down to a surface $X_{\mathbb{R}}$ such that E (resp. E and $\sigma(E)$) is contracted to a smooth real (resp. complex conjugate) point(s) P and $Y_{\mathbb{R}}$ is precisely the blow-up of $X_{\mathbb{R}}$ at the point P as described above.

A *real del Pezzo surface* $X_{\mathbb{R}}$ is by definition a connected surface with a real structure σ that is possibly singular but Gorenstein and is such that $-K$ is ample. In the complex case, we have already seen that the smooth del Pezzo surface $X_{\mathbb{C}}$ are $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2_r$ obtained by blowing up r points in general positions on the projective surface $\mathbb{CP}^2$. The roots of $Pic(X_{\mathbb{C}})$, defined as rational divisors orthogonal to $-K$, form a Dynkin diagram and generate a Weyl group W. It can be proved that W is the group of isometries of $Pic(X_{\mathbb{C}})$, leaving K fixed [12, 13]. Then, the real structure σ on $X_{\mathbb{R}}$ belongs to a conjugacy class of W. We use σ to decompose the root lattice L into eigenspaces $L^+ \oplus L^-$. It is known that σ is a product of $\dim L^-$ reflections and the subgroup W_σ of W leaving L^+ pointwise fixed is itself a group generated by reflexions. We write $\triangle(\sigma)$ for its Dynkin diagram. The real structure σ is entirely characterized by $\triangle(\sigma)$ (table 2) [14]. We note that if we start with a real del Pezzo surface $(X^r_{\mathbb{R}}, \sigma)$ and blow up one real point P (resp. complex conjugate points P_1 and P_2), we obtain a real del Pezzo surface $(X^{r+1}_{\mathbb{R}}, \sigma')$ with $\triangle(\sigma') = \triangle(\sigma)$ (resp. $\triangle(\sigma') = \triangle(\sigma) \times A_1$ whose new root $E_1 - E_2$ is anti-invariant under σ').

We can now write down explicitly the involutions appearing in table 3. We have already seen that the involution A_1^r correspond to blow up r pairs of complex conjugate points. If $d = 8$, A_1' corresponds to $\sigma_{A_1'}(E_i) = E_i + E_0 - E_1 - E_2 - E_3$. $A_1^{2'}$ $(d = 6)$ and $A_1^{3'}$ $(d = 4)$

$d=K^2{+}2$	$Weyl(\mathbb{CP}^2_{11-d})$											
11	$+$	1										
10	$+$	1										
9	A_1	1	A_1									
8	$A_1\times A_2$	1	A_1			$A_1^{'}$	$D_2\simeq A_1^2$					
7	A_4	1	A_1	A_1^2								
6	D_5	1	A_1	A_1^2		$A_1^{2'}$	$D_2\times A_1$	D_4				
5	E_6	1	A_1	A_1^2	A_1^3			D_4				
4	E_7	1	A_1	A_1^2	A_1^3	$A_1^{3\prime}$	$D_2\times A_1^2$	D_4	$D_4\times A_1$	D_6	E_7	
3	E_8	1	A_1	A_1^2	A_1^3		A_1^4	D_4	$D_4\times A_1$	D_6	E_7	E_8

Table 3: Conjugacy classes of involutions, $\triangle(\sigma)$ Weyl subgroups of $\mathbb{CP}^2_{11-d}$.

are now obtained from $A_1^{'}$ by blowing up one or two pairs of complex conjugate points. The involution $A_1^{'}$ can be put in a more standard way by blowing down the (-1)-curves $H-E_1-E_2$ and E_3 exchanged by $\sigma_{A_1^{'}}$. We obtain a del Pezzo surface whose Picard group is generated by the lines $l_1 := H-E_1$ and $l_2 := H-E_2$, it is invariant under $\sigma_{A_1^{'}}$. This surface corresponds to the real form of the del Pezzo surface $\mathbb{CP}^1\times\mathbb{CP}^1$ with the trivial involution 1. It is a quadric hypersurface $x_1^2+x_2^2-x_3^2-x_4^2=0$, denoted $Q^{(2,2)}$. So, the involution A_1' can be obtained by blowing up two complex conjugate points on $Q^{(2,2)}$. If we blow up a real point on $Q^{(2,2)}$, we obtain a surface isomorphism to $\mathbb{CP}^2$ with the trivial involution and blown up on two real points. A second real form of the del Pezzo surface $\mathbb{CP}^1\times\mathbb{CP}^1$ can be obtained by imposing an A_1 involution given by $\sigma_{A_1}(l_1)=l_2$ and we have the quadric hypersurface $x_1^2+x_2^2+x_3^2-x_4^2=0$, denoted $Q^{(3,1)}$. This surface, blown up on a real point, is isomorphic to $\mathbb{CP}^2$ with the trivial involution and blown up on two complex conjugate points.

The involution D_{2s} is

$$\sigma(H) = (s-1)(H-E_1)-K-H \tag{3.1}$$

$$\sigma(E_1) = -E_1-K+(s-3)(H-E_1) \tag{3.2}$$

$$\sigma(E_i) = H-E_1-E_i \qquad (2\le i\le 2s+1)\,. \tag{3.3}$$

The involution E_7 (resp. E_8) correspond to the Geiser (resp. Bertini) involution and acts on $Pic(\mathbb{CP}^2_7)$ (resp. $Pic(\mathbb{CP}^2_8)$) as $\sigma(D)=(D.K)K-D$ (resp. $\sigma(D)=2(D.K)K-D$).

4. Real forms of Lie algebras

In this section, we will explain briefly how to classify the *real Lie algebras* according to the *Tits-Satake diagram.* A real form of a complex Lie algebra $\mathcal{G}^{\mathbb{C}}$ is defined by the restriction of the field of coefficients from the complex to the real numbers. In M-theory, we must eventually replace the real field by the integers. For example, the choice of real coefficients in the Cartan-Weyl basis defines the split form and its existence follows from the fact that the structure constants of $\mathcal{G}^{\mathbb{C}}$ can be taken to be integers. For $\mathrm{SL}(2,\mathbb{C})$, the split form is $\mathrm{SL}(2,\mathbb{R})$ wheras the compact form is $\mathrm{SU}(2)$ and the arithmetic group is the famous modular group $\mathrm{SL}(2,\mathbb{Z})$. These diagrams of the real forms of the classical and exceptional complex groups have been classified by Araki [15].

JHEP04(2003)060

Let $\mathcal{G}$ be a real simple Lie algebra with Cartan decomposition $\mathcal{G} = \mathcal{T} \oplus^{\perp} \mathcal{P}$ with $\mathcal{T}$ a maximal compact subalgebra of $\mathcal{G}$ and $\mathcal{P}$ the orthogonal complement of $\mathcal{T}$ with respect to the Killing form which is negative definite when restricted to $\mathcal{T}$. We can define a linear Cartan involution θ by $\theta \mid_{\mathcal{P}} = -1$ and $\theta \mid_{\mathcal{T}} = 1$. For the split form, the Cartan-Chevalley involution θ is given by $\theta(h_\alpha) = -h_\alpha$, $\theta(e_\alpha) = -e_{-\alpha}$. Then, the maximal compact subalgebra $\mathcal{T}$ is generated over $\mathbb{R}$ by $\{e_\alpha - e_{-\alpha}\}$ and $\mathcal{P}$ by $\{h_\alpha\}$ and $\{e_\alpha + e_{-\alpha}\}$.

Let $\mathcal{A}$ be a maximal abelian subalgebra of $\mathcal{P}$, and let η be (a maximally non compact θ-stable) Cartan subalgebra containing $\mathcal{A}$ ($\mathcal{A}^*$ is its dual). The dimension of $\mathcal{A}$ corresponds to the real rank of $\mathcal{G}$. For the split form, $\mathcal{A}$ is generated over $\mathbb{R}$ by the Cartan generators $\{h_\alpha\}$ and the real rank coincides with the rank of its complexification. The conjugation σ of $\mathcal{G}^{\mathbb{C}}$, the complexification of $\mathcal{G}$, with respect to $\mathcal{G}$ defines an involutive automorphism and all the fixed elements of $\mathcal{G}^{\mathbb{C}}$ by σ are the real form $\mathcal{G}$. Thus, we have $\sigma(X) = X$ and $\sigma(iX) = -iX \ \forall X \in \mathcal{G}$. In the same way, we can define the conjugation $\tau = \theta.\sigma$ of $K\mathcal{G}^{\mathbb{C}}$ with respect to the compact real form $\mathcal{G} = \mathcal{T} \oplus i\mathcal{P}$. The Satake diagram permits to obtain easily the conjugation σ and then to find the real form associated to it.

The Satake diagram of $(\mathcal{G}, \mathcal{A})$ consists of

1. The Dynkin diagram of $(\mathcal{G}^{\mathbb{C}}, \eta^{\mathbb{C}})$. The root lattice is noted Δ.

2. A coloring of the vertices of the diagram: black if the associated simple root of $(\mathcal{G}^{\mathbb{C}}, \eta^{\mathbb{C}})$ restricts to 0 on $\mathcal{A}^*$, white otherwise. Let $\bar{\alpha} = \frac{1}{2}(\alpha - \theta(\alpha))$ denote the restriction of $\alpha \in \Delta$ to the subspace $\mathcal{A}^*$ and $m_\psi = \dim\{\alpha \in \Delta \mid \bar{\alpha} = \psi\}$, called the multiplicity of ψ. We set $\Delta_0 = \{\alpha \in \Delta : \bar{\alpha} = 0\}$ and $\Delta_1 = \Delta - \Delta_0$.

3. A "curved arrow" joining two white vertices if and only if the associated simple roots α and β restrict to the same root on $\mathcal{A}^*$: $\bar{\alpha} = \bar{\beta}$.

4. The Dynkin diagram of the root system: $\Sigma = \{\bar{\alpha} : \alpha \in \Delta_1\}$.

A table of Satake diagrams of the real simple Lie algebras can be found in Helgason [16]. The involution σ can be determined by the Satake diagram of $\mathcal{G}$. Indeed, it can be proved that we can choose a basis of $\eta^{\mathbb{C}}$ such as $\tau(\alpha) = -\alpha$ and so $\sigma = -\theta$ on the root lattice. So, if the vertex α_i in the Satake diagram of $\mathcal{G}$ is black, the action of σ is $\sigma(\alpha_i) = -\alpha_i$ and if two white vertices α_i and α_j are joined by a curved arrow, we have $\sigma(\alpha_i) + \alpha_i = \sigma(\alpha_j) + \alpha_j$. Then, the action of σ on the other roots can be deduced by using the facts that σ is an involution which must preserve the Cartan matrix. More precisely, let $\Delta_1 = \{\alpha_1, \cdots, \alpha_r\}$ and $\Delta_0 = \{\alpha_{(r+1)}, \cdots, \alpha_s\}$, then $\sigma(\alpha_i) = \alpha_{\pi(i)} + \sum_{i=r+1}^{s} c_{ij}\alpha_j$ where π is an involutive permutation of $\{1, 2, \cdots, r\}$ and c_{ij} are non-negative integers. The proof is very simple: let us decompose $\sigma(\alpha_i) = \sum m_{ij}\alpha_j + \sum_{i=r+1}^{s} c_{ij}\alpha_j$ on the basis Δ. Now applying σ to this identity, we have $\sum_j m_{ij}m_{jk} = \delta_{ik}$ which means that m_{ij} can be considered as the elements of the permutation matrix and we obtain the desired result. For example, if $\mathcal{G}$ is the split form, σ is the identity. For the non-split $E_{7|-5}$ of real rank 4, we give as an exercice the involution σ (table 4). The associated restricted root system of Σ is that of its maximal split subalgebra F_4 [17].

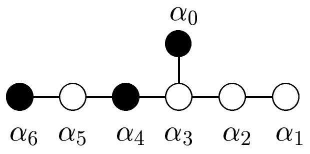
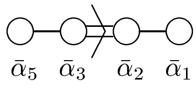
Δ	Σ	σ
	$m(2\bar{\alpha}_i) = 1$ $m(\bar{\alpha}_i) = 1$ 	$\sigma(\alpha_0) = -\alpha_0$ $\sigma(\alpha_1) = \alpha_1$ $\sigma(\alpha_2) = \alpha_2$ $\sigma(\alpha_3) = \alpha_3 + \alpha_0 + \alpha_4$ $\sigma(\alpha_4) = -\alpha_4$ $\sigma(\alpha_5) = \alpha_5 + \alpha_4 + \alpha_6$ $\sigma(\alpha_6) = -\alpha_6$

Table 4: Satake diagram of $E_{7|-5}$.

5. Real magic triangle/square

5.1 Origins

In [3], the *oxidation endpoint* of the pure supergravity with $\mathcal{N}$ supersymmetries in four dimensions was obtained. By the oxidation endpoint, we mean the supergravity theory in the highest possible dimension whose toroidal dimensional reduction gives back precisely to the pure supergravity in four dimensions with $\mathcal{N}$ SUSY spinors. For example, the pure supergravity $\mathcal{N} = 8$ in $d = 4$ can be obtained by dimensional reduction of eleven-dimensional supergravity [18]. The oxidation sequence is presented in table 5 and we have indicated the corresponding U-duality cosets $G/K_G \,|^{\dim G}_{\dim K_G}$. G is a real Lie group and K_G, which corresponds to the R-symmetry, is the maximal compact Lie group of G. A similar magic triangle (table 6), corresponding to Maxwell-Einstein supergravities $\mathcal{N} = 2$ $d = 5$, has be obtained by [4]. The following U-duality cosets can be guessed simply by checking that the dimension of the cosets equals the number of scalars. We recall that in order to obtain these U-duality groups, some forms must be dualized. In the following section, we will double all field strengths with their duals and we will infer an even larger symmetry, which corresponds to a *real form of a Borcherds superalgebra.* The SM$\triangle$ and M$\triangle$ have also recently been studied [19, 20] using group theory.

The oxidation sequence of each column $\mathcal{N}$ can be obtained by deleting a line of white roots at the end of the extended Dynkin-Satake diagrams where the affine vertex attaches [21]. We already mentioned the mysterious mechanism of disintegration of the R-symmetry that starts also from the affine end but of the maximal compact U-duality Dynkin diagrams. Alternatively we can obtain the non-split U-duality algebras of each column of the Magic Triangle starting from the *Vogan diagram* of $E_{n|n}$. The Vogan diagrams encode the real forms of Lie algebras using a maximal compact Cartan subalgebra [22]. The Vogan bicoloured diagrams may be chosen to have all or all but one compact vertices, represented now by white dots. Let us take the example of $n = 8$. The white dots with the affine root generate the Dynkin diagram of the maximal compact algebra SO(16) and the SUSY breaking sequence in $d = 3$ amounts to the deletion of white dots starting from the affine root of E_8.

	$\mathcal{N}=7$	$\mathcal{N}=6$	$\mathcal{N}=5$	$\mathcal{N}=4$	$\mathcal{N}=3$	$\mathcal{N}=2$	$\mathcal{N}=1$	$\mathcal{N}=0$
$d=11$	$+$							
$d=10$	$\mathbb{R}\,\vert_{0}^{1}$							
$d=9$	$\frac{\mathrm{SL}(2)}{\mathrm{SO}(2)}\times\mathbb{R}\,\vert_{1}^{4}$							
$d=8$	$\frac{\mathrm{SL}(3)\times\mathrm{SL}(2)}{\mathrm{SO}(3)\times\mathrm{SO}(2)}\,\vert_{4}^{11}$							
$d=7$	$\frac{\mathrm{SL}(5)}{\mathrm{SO}(5)}\,\vert_{10}^{24}$							
$d=6$	$\frac{\mathrm{SO}(5,5)}{\mathrm{SO}(5)\times\mathrm{SO}(5)}\,\vert_{20}^{45}$	$\frac{\mathrm{SO}(5,1)\times\mathrm{SO}(3)}{\mathrm{SO}(5)\times\mathrm{SO}(3)}\,\vert_{13}^{18}$						
$d=5$	$\frac{E_6}{Usp(8)}\,\vert_{36}^{78}$	$\frac{SU^\star(6)}{Usp(6)}\,\vert_{21}^{35}$						
$d=4$	$\frac{E_7}{\mathrm{SU}(8)}\,\vert_{63}^{133}$	$\frac{SO^\star(12)}{\mathrm{U}(6)}\,\vert_{36}^{66}$	$\frac{\mathrm{SU}(5,1)}{\mathrm{U}(5)}\,\vert_{25}^{35}$	$\frac{\mathrm{SU}(4)\times\mathrm{SU}(1,1)}{\mathrm{SU}(4)\times\mathrm{SO}(2)}\,\vert_{16}^{18}$	$\frac{\mathrm{U}(3)}{\mathrm{U}(3)}$	$\frac{\mathrm{U}(2)}{\mathrm{U}(2)}$	$\frac{\mathrm{U}(1)}{\mathrm{U}(1)}$	$+$
$d=3$	$\frac{E_8}{\mathrm{SO}(16)}\,\vert_{120}^{248}$	$\frac{E_7}{S0(12)\times S0(3)}\,\vert_{69}^{133}$	$\frac{E_6}{\mathrm{SO}(10)\times\mathrm{SO}(2)}\,\vert_{46}^{78}$	$\frac{\mathrm{SO}(8,2)}{\mathrm{SO}(8)\times\mathrm{SO}(2)}\,\vert_{29}^{45}$	$\frac{\mathrm{SU}(4,1)}{\mathrm{SO}(6)\times\mathrm{SO}(2)}\,\vert_{24}^{16}$	$\frac{\mathrm{SU}(2,1)\times\mathrm{SU}(2)}{\mathrm{SO}(4)\times\mathrm{SO}(2)}\,\vert_{7}^{11}$	$\frac{\mathrm{SL}(2)\times\mathrm{SO}(2)}{\mathrm{SO}(2)\times\mathrm{SO}(2)}\,\vert_{2}^{4}$	$\frac{\mathrm{SL}(2)}{\mathrm{SO}(2)}\,\vert_{1}^{3}$

Table 5: Real magic triangle Cosets.

$d=6$	$\frac{\mathrm{SO}(9,1)}{\mathrm{SO}(9)}\Big\vert^{45}_{37}$	$\frac{\mathrm{SO}(5,1)\times\mathrm{SO}(3)}{\mathrm{SO}(5)\times\mathrm{SO}(3)}\Big\vert^{18}_{13}$	$\frac{\mathrm{SL}(2,\mathbb{C})\times\mathrm{U}(1)}{\mathrm{SU}(2)\times\mathrm{U}(1)}\Big\vert^{8}_{4}$
$d=5$	$\frac{E_6}{F_4}\Big\vert^{78}_{52}$	$\frac{SU^*(6)}{Usp(6)}\Big\vert^{35}_{21}$	$\frac{\mathrm{SL}(3,\mathbb{C})}{\mathrm{SU}(3)}\Big\vert^{17}_{8}$
$d=4$	$\frac{E_7}{E_6\times\mathrm{U}(1)}\Big\vert^{133}_{69}$	$\frac{SO^*(12)}{\mathrm{U}(6)}\Big\vert^{66}_{36}$	$\frac{\mathrm{SU}(3,3)}{S(\mathrm{U}(3)\times\mathrm{U}(3))}\Big\vert^{35}_{17}$
$d=3$	$\frac{E_8}{E_7\times\mathrm{SU}(2)}\Big\vert^{248}_{136}$	$\frac{E_7}{\mathrm{SO}(12)\times\mathrm{SU}(2)}\Big\vert^{133}_{79}$	$\frac{E_6}{\mathrm{SU}(6)\times\mathrm{SU}(2)}\Big\vert^{78}_{38}$

Table 6: Magic square.

5.2 Construction with real singularities

We proved [2] that the field theories of the SM△ correspond to normal del Pezzo surfaces $X_{\mathbb{C}}$ with Du Val singularities resulting from the contraction of a set of intersecting (-2)-curves forming a Dynkin diagram of type A_* ie $Sl(n)$. In order to obtain other real forms of the U-duality groups than the split ones, it seems natural to guess that the supergravity theories of the original M△ correspond to real normal del Pezzo surfaces $X_{\mathbb{R}}$ with real (Du Val-type) singularities which are associated to real forms of A_*. The real forms of A_* that can be obtained are $Sl(n)$, $\mathrm{SU}(n+1,n-1)$, $\mathrm{SU}(n,n)$ and $\mathrm{SU}(n,n-1)$ with the Tits-Satake diagrams [23, 24] of table 7.

Then, the supergravity theory $(\mathcal{N},d)$ of M△ in d dimensions with $\mathcal{N}$ SUSY is identified with the normal real del Pezzo surface $X_{\mathbb{R}}$ with $d = K^2+2$ and a real singularity $A_{(8-\mathcal{N})}$ in the following way: the Satake involution which acts on the root lattice L of the Dynkin diagram of the U-duality group G is identified with the real structure σ of the real surface $X_{\mathbb{R}}$. By using the facts that $Pic(X_{\mathbb{R}}) = L \oplus \mathbb{Z}K$ and $\sigma(K) = K$ we deduce the action of the real structure σ on the Picard group of $X_{\mathbb{R}}$.

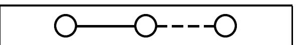
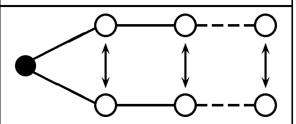
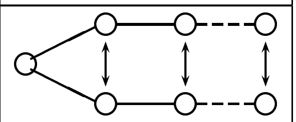
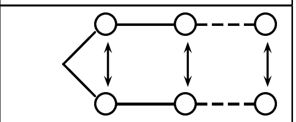
$\mathrm{SL}(n)$	
$\mathrm{SU}(n+1,n-1)$ $n\geq 1$	
$\mathrm{SU}(n,n)$ $n\geq 1$	
$\mathrm{SU}(n,n-1)$ $n\geq 1$	

Table 7: Satake diagrams of A_{n-1}.

Now, the $A_{(8-\mathcal{N})}$ singularity, characterized by another Satake involution σ_A, remains to be determined. The Picard group of the normalized real del Pezzo surface $\tilde{X}_{\mathbb{R}}$, which is isomorphic in the complex case to $\mathbb{CP}^2_{11-d}$, is given by $Pic(\tilde{X}_{\mathbb{R}}) = Pic(X_{\mathbb{R}}) \oplus^{\perp} A_{(8-\mathcal{N})}$. So the real structure of $\tilde{X}_{\mathbb{R}}$ is $\tilde{\sigma} = \sigma \oplus \sigma_A$ and must correspond to those listed in table 3. In some particular cases, we will obtain more than one solution. In the following table (table 8), the real normal del Pezzo surface $X_{\mathbb{R}}$, corresponding to the supergravity $(\mathcal{N},d)$ of the M△, is noted $(\Delta(\tilde{\sigma}), A)$ where A is a real form of the $A_{(8-\mathcal{N})}$ singularity. A similar table (table 9) can be obtained for the supergravity theories of the magic square. It should be noted that the link between the number of singularities and the number of supersymmetries is not obvious as each theory of the magic square corresponds to $\mathcal{N}=2$ or more.

	$\mathcal{N}=7$	$\mathcal{N}=6$	$\mathcal{N}=5$	$\mathcal{N}=4$	$\mathcal{N}=3$	$\mathcal{N}=2$	$\mathcal{N}=1$	$\mathcal{N}=0$
$d=11$	$+$							
$d=10$	$(1,0)$							
$d=9$	$(1,0)$							
$d=8$	$(1,0)$							
$d=7$	$(1,0)$							
$d=6$	$(1,0)$	$(A_1^3, \mathrm{SU}(1,1))$ $(D_4, \mathrm{SU}(1,1))$						
$d=5$	$(1,0)$	$(A_1^3, \mathrm{SU}(1,1))$ $(D_4, \mathrm{SU}(2))$						
$d=4$	$(1,0)$	$(A_1^3, \mathrm{SU}(1,1))$ $(D_4, \mathrm{SU}(2))$	$(A_1 \times D_4, \mathrm{SU}(2,1))$	$(A_1 \times D_4, \mathrm{SU}(3,1))$ $(D_4, \mathrm{SU}(2,2))$	$(A_1 \times D_4, \mathrm{SU}(3,2))$	$(A_1 \times D_4, \mathrm{SU}(4,2))$ $(A_1^4, \mathrm{SU}(3,3))$	$(A_1^4, \mathrm{SU}(4,3))$	$(1, \mathrm{SL}(8))$ $(A_1^3, \mathrm{SU}(4,4))$ $(D_4, \mathrm{SU}(5,3))$
$d=3$	$(1,0)$	$(A_1^3, \mathrm{SU}(1,1))$ $(D_4, \mathrm{SU}(2))$	$(A_1 \times D_4, \mathrm{SU}(2,1))$	$(A_1 \times D_4, \mathrm{SU}(3,1))$ $(D_4, \mathrm{SU}(2,2))$	$(A_1 \times D_4, \mathrm{SU}(3,2))$	$(A_1 \times D_4, \mathrm{SU}(4,2))$ $(A_1^4, \mathrm{SU}(3,3))$	$(A_1^4, \mathrm{SU}(4,3))$	$(1, \mathrm{SL}(8))$ $(A_1^3, \mathrm{SU}(4,4))$ $(D_4, \mathrm{SU}(5,3))$

Table 8: Real magic del Pezzo's triangle: real structure on the resolution (given by the anti-invariant subdiagram) and real form of the singularity.

$d=5$	$(D_4,0)$	$(A_1^3,\mathrm{SL}(2))$ $(D_4,\mathrm{SU}(2))$	$(A_1^3,\mathrm{SU}(2,1))$
$d=4$	$(D_4,0)$	$(A_1^3,\mathrm{SL}(2))$ $(D_4,\mathrm{SU}(2))$	$(A_1^3,\mathrm{SU}(2,1))$
$d=3$	$(D_4,0)$	$(A_1^3,\mathrm{SL}(2))$ $(D_4,\mathrm{SU}(2))$	$(A_1^3,\mathrm{SU}(2,1))$

Table 9: Real magic del Pezzo's square.

We observe that, as in the complex case [2], each vertical step down, which corresponds on the supergravity side to a compactification on a circle, corresponds to blowing up one real point. The singularity A, $\Delta(\sigma)$ and $\Delta(\tilde{\sigma})$ are unchanged under this operation.

5.3 Real Borcherds superalgebra

We will show in this section how the connection between supergravity theories and real del Pezzo surfaces can be useful to show that the U-duality algebra can be enlarged into a real Borcherds superalgebra. The basic procedure is a direct extension of [2]. According to this paper, we can associate to a complex del Pezzo surface $X_{\mathbb{C}}$ a Borcherds superalgebra whose simple roots α_i generate the full Picard group $Pic(X_{\mathbb{C}})$ and the set of positive roots contains the rational (i.e of vanishing virtual genus) divisor classes of non negative degree and also (except in the case of M-theory) the anticanonical class $-K$.

We can then define minus the intersection matrix $A_{ij} = -\alpha_i.\alpha_j$ and a $\mathbb{Z}_2$-graduation by $\mathrm{grad}(\alpha_i) = -K.\alpha_i \; mod\, 2$. However, it turns out that whenever a fermionic root of square -1 appears it should be viewed as an $sl(1|1)$ superroot, i.e. it should have zero Cartan-Killing norm. The corresponding modified matrix will be our *Cartan matrix* a_{ij}. This matrix a_{ij} satisfies the following properties and thus defines a Borcherds superalgebra (or *Generalized Kac-Moody* superalgebra).

$$\text{(i)} \quad a_{ij} \leq 0 \qquad if \; i \neq j \tag{5.1}$$

$$\text{(ii)} \quad \frac{2a_{ij}}{a_{ii}} \in \mathbb{Z} \qquad if \; a_{ii} > 0\,, \mathrm{grad}(\alpha_i) = 0 \tag{5.2}$$

$$\text{(iii)} \quad \frac{a_{ij}}{a_{ii}} \in \mathbb{Z} \qquad if \; a_{ii} > 0\,, \mathrm{grad}(\alpha_i) = 1\,. \tag{5.3}$$

The Borcherds superalgebra associated to the matrix a_{ij} has a Cartan subalgebra H, with basis $\{h_{\alpha_i}\}$, it is by definition the Lie superalgebra G generated by H and by the elements e_{α_i}, f_{α_i} satisfying here the following elementary relations and their consequences:

$$(1) \quad [e_{\alpha_i}, f_{\alpha_j}] = \delta_{ij} h_{\alpha_i} \tag{5.4}$$

$$(2) \quad [h_{\alpha_j}, e_{\alpha_i}] = a_{ij} e_{\alpha_i}, [h_{\alpha_j}, f_{\alpha_i}] = -a_{ij} f_{\alpha_i} \tag{5.5}$$

$$(3) \quad [h_{\alpha_i}, h_{\alpha_j}] = 0 \tag{5.6}$$

$$(4) \quad ad(e_{\alpha_i})^{1-2\frac{a_{ij}}{a_{ii}}} e_{\alpha_j} = 0 = (ad(f_{\alpha_i}))^{1-2\frac{a_{ij}}{a_{ii}}} f_{\alpha_j} \text{ if } a_{ii} > 0 \tag{5.7}$$

$$(5) \quad [e_{\alpha_i}, e_{\alpha_j}] = 0 = [f_{\alpha_i}, f_{\alpha_j}] \text{ if } a_{ij} = 0\,. \tag{5.8}$$

A Killing form is defined as $B(e_\alpha, e_{-\alpha}) = -1$ and $B(h, h_\alpha) = \alpha(h) \; \forall\, h \in H$.

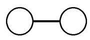
Δ	Σ	σ
 α_3 α_2 α_1 β α_0	 $\bar{\alpha_2}$ $\bar{\beta}$ **4** **4**	$\sigma(\alpha_i) = -\alpha_i \,\forall i = 0,1,2$ $\sigma(\alpha_2) = \alpha_2 + \alpha_3 + \alpha_1$ $\sigma(\beta) = \beta + \alpha_0 + \alpha_1$ $\bar{a} = \begin{pmatrix} 2 & -1 \\ -1 & -2 \end{pmatrix}$

Table 10: Real form of the Borcherds superalgebra B_6^6 of $\mathcal{N} = 6$, $d = 6$ supergravity.

Using this contruction, we can associate to a del Pezzo surface $X_{\mathbb{R}}$ with real structure σ a real Borcherds superalgebra $\mathcal{G}$ with a conjugation given by σ. However, it does not work for all real structures on a given del Pezzo, because for real superalgebras, one further imposes that the σ-system of roots satisfies the *normal condition* [15] which means that for any bosonic root α, $\alpha - \sigma(\alpha)$ must not be a root. Following Araki, we can define *Satake superdiagrams* (appendix) which recently appeared for the classical Lie superalgebras case in [25]. All the real Borcherds superalgebras which will appear, have only fermionic imaginary simple roots, of vanishing norm, noted β_i. The other bosonic roots which generate the U-duality algebra will be noted α_i. We note that the restricted roots Σ form a Borcherds superalgebra $\bar{\mathcal{G}}$ and its Cartan matrix is noted $\bar{a}_{ij}$. Then, we color the superdiagrams as explained is section 4 (this convention differs from that of our previous paper). For example, the Borcherds superalgebra B_6^6 of the supergravity $\mathcal{N} = 6$ and $d = 6$ is given in table 10.

In order to obtain the equations of motion of the supergravity theory and to show that they are invariant under the (truncated) Borel supergroup of a real Borcherds superalgebra $\mathcal{G}$, we need to generalize the Iwasawa decomposition which can be studied in [16]. The proof remains the same and the decompositon reads as $\mathcal{G} = \mathcal{T} + \mathcal{A} + \mathcal{N}$ where $\mathcal{N} \equiv \sum_{\bar{\lambda} \in \Sigma^+} \mathcal{G}_{\bar{\lambda}}$ is the nilpotent subalgebra of positive restricted roots. The other notations are those used in section 2. We note that $\dim(\mathcal{G}_{\bar{\lambda}}) = m(\bar{\lambda}).mult(\bar{\lambda})$ where $mult(\bar{\lambda})$ is the mutiplicity of the $\bar{\lambda}$ for the Borcherds superalgebra $\bar{\mathcal{G}}$. Let us introduce the following nonlinear "potential" differential form:

$$\mathcal{V} = \prod_{\bar{\lambda} \in \mathcal{A}^*} e^{h_{\bar{\lambda}} \phi_{\bar{\lambda}}} \prod_{\bar{\lambda} \in \Sigma^+ \,|\, d(\bar{\lambda}) \leq D} e^{e_{\bar{\lambda}} A_{\bar{\lambda}}} \,. \tag{5.9}$$

The Grassman real angle $A_{\bar{\lambda}}$ is a form of degree $d(\bar{\lambda}) \equiv -K.\bar{\lambda}$ lower than $d \equiv K^2 + 2$ ($\phi_{\bar{\lambda}}$ is 0-form) defined on a formal manifold of dimension d. The field strength $\mathcal{G} \equiv d\mathcal{V}\mathcal{V}^{-1}$ is then a element of $\mathcal{N}$ whose coefficents are given by forms. As in [2], we introduce the pseudo-involution $\mathcal{S}$ satisfying:

$$\mathcal{S}(e_{\bar{\lambda}}) = e_{-\bar{\lambda}-K} \,\forall \bar{\lambda} \in \Sigma^+ \;|\; d(\bar{\lambda}) \leq d \,.$$

This operator is well defined because the truncated root lattice of the Borcherds superalgebra is invariant under this involution. Finally, if we combine the (twisted) self-duality equations $S\mathcal{G} = *\mathcal{G}$ and the Maurer-Cartan equation $d\mathcal{G} = -\mathcal{G} \wedge \mathcal{G}$ which follows by taking the exterior derivative of the field strength, we obtain some second-order equations which reproduce the equations of motion for the p-forms of the supergravity theory with

JHEP04(2003)060

Degree	positive root	mutiplicity	potential
0	0		ϕ
0	$\bar{\alpha}_2$	4	$A_{(0)}$
1	$\bar{\beta}$	4	$A_{(1)}$
1	$\bar{\alpha}_2+\bar{\beta}$	4	$\mathcal{A}_{(1)}$
2	$2\bar{\beta}$	1	$A_{(2)}$
2	$\bar{\alpha}_2+2\bar{\beta}$	1	$\mathcal{A}_{(2)}$
2	$2(\bar{\alpha}_2+\bar{\beta})$	1	$\widetilde{A}_{(2)}$
3	$\bar{\alpha}_2+3\bar{\beta}$	4	$\widetilde{\mathcal{A}}_{(3)}$
3	$2\bar{\alpha}_2+3\bar{\beta}$	4	$\widetilde{A}_{(3)}$
4	$\bar{\alpha}_2+4\bar{\beta}$	4	$A_{(4)}$
4	$3\bar{\alpha}_2+4\bar{\beta}$	4	none
4	$2\bar{\alpha}_2+4\bar{\beta}$	1	$-K$

Table 11: Restricted roots of B_6^6 with their multiplicities, and associated potentials.

the corresponding real finite simple Lie algebra (*i.e.* U-duality algebra) . These equations are trivially invariant under the Borel supergroup whose Lie superalgebra is generated by $\mathcal{A}+\mathcal{N}$. Indeed we can define a natural action on the generalized potential $\mathcal{V}$ by $\mathcal{V}'=\mathcal{V}.\Lambda$ with Λ an element of the Borel supergroup with coefficients given by closed forms. The field strength $\mathcal{G}$ is then invariant. Table 11 shows the positive restricted roots of B_6^6 with their multiplicities.

The root $3\bar{\alpha}_2+4\bar{\beta}$ is exchanged under the pseudo-involution $\mathcal{S}$ with the negative root $-\bar{\alpha}_1$, which corresponds to the NS-1 instanton which is S-dual to the D-1 instanton coupled to the scalar $A_{(0)}$. The root $3\bar{\alpha}_2+4\bar{\beta}$ is therefore the Hodge-dual of NS-1 and could be noted NS3. We can check that we obtain the right number of fields ie four axions, eight 1-forms, three 2-forms and the corresponding dual fields.

6. Conclusion

Recently, some other conjectured symmetries of M-theory such as Kac-Moody algebras E_{10} and E_{11}, introduced a long time ago [26] for the first one (and partly realised [28]) and more recently [27] for the second one and further studied in [29] have reappeared. The hyperbolic Kac-Moody algebra E_{10} appears when one studies the chaotic behaviour of the M-theory near a cosmological singularity. The solutions are chaotic if the corresponding Kac-Moody algebra is hyperbolic which means that one obtains some finite dimensional simple Lie algebras or affine Lie algebras by deleting a simple root to its Dynkin diagram [30]. For the supergravity theory with $\mathcal{N}$ supersymmetries in M$\triangle$, the corresponding over-extension can be found using the corresponding Borcherds superalgebra in $d=3$ dimensions. As noted in [2], the fermionic simple root β stands at the location of the affine root which is invariant under the Satake conjugation. The over-extension corresponds to add two invariant (white) simple roots starting with the affine root. On the del Pezzo side, it corresponds to blowing up two more real points.

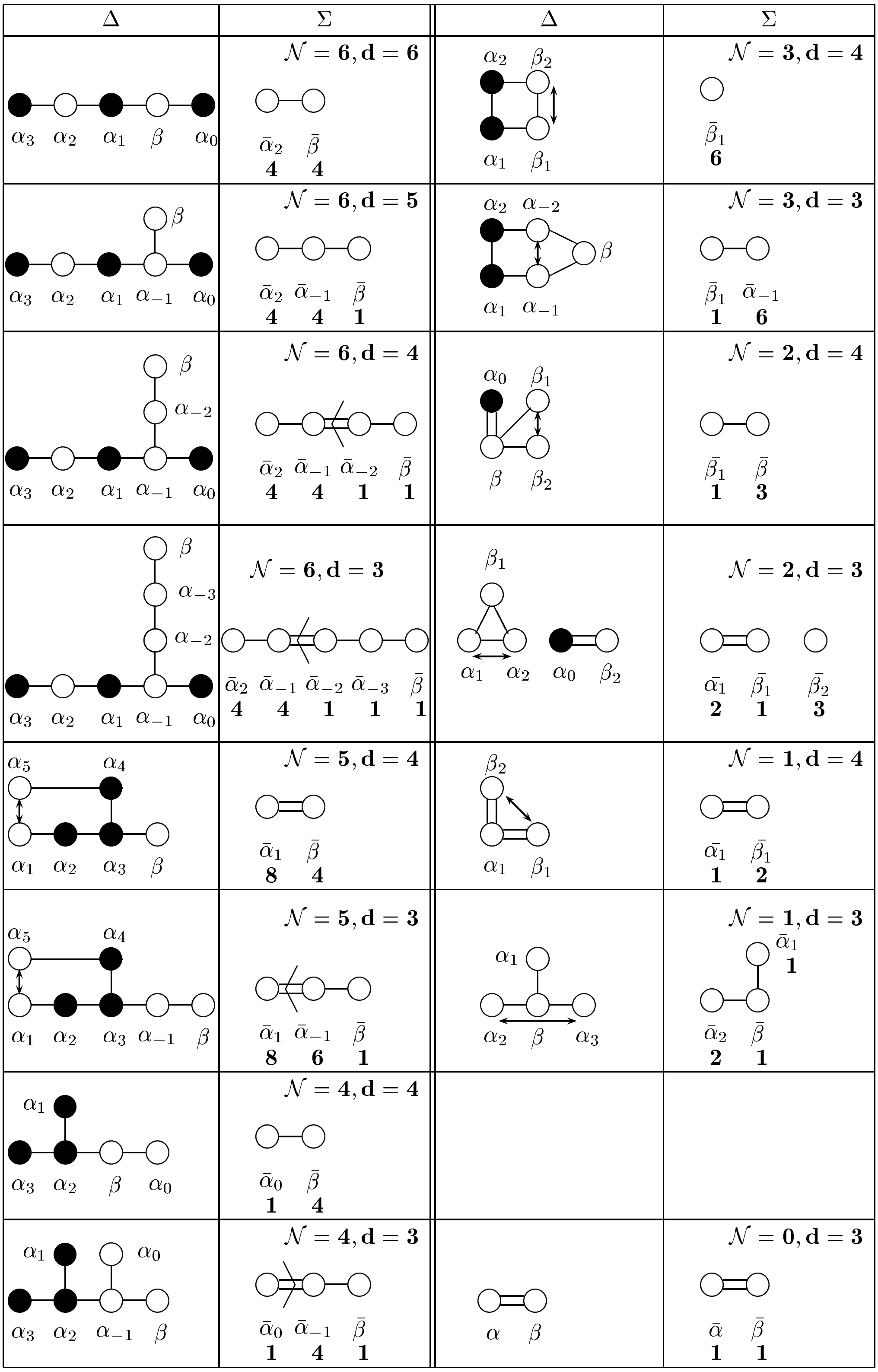


Table 12: Satake diagrams of superalgebras of the magic triangle and restricted roots.

These algebras require also a new formulation of M-theory which incorporates the fields and their duals. However, in order to obtain these algebras, we need to dualize also the gravity field. Our supercoset construction based on some real Borcherds superalgebra must be extended in order to include these fields which can be described by multiforms [31, 32]. For example, the gravity field and its dual are described by bi-forms. We are witnessing indeed magical coincidences with some defects that should be corrected to reach harmony and that in fact suggest a bigger structure yet to be discovered. In particular the role of SUSY is still a puzzle.

A. Real superalgebras of the real triangle

In table 12, we list the real Borcherds superalgebra corresponding to each box of the SMΔ. We recall that β_i is a fermionic root of vanishing norm and α_j is a bosonic root, normalized such that ${\alpha_j}^2 = 2$. $\bar{}$ corresponds to the projection under $\mathcal{A}$, the maximal abelian subalgebra of $\mathcal{P}$ (cf section 4). The bold number below a reduced simple root $\bar{\lambda}$ correspond to its multiplicity $\dim(\mathcal{G}_{\bar{\lambda}})$. We note that in $d = 3$ dimensions, the fermionic simple root of the reduced Borcherds superalgebra stands at the location of the affine root. This is not so well defined for semi-simple non-simple Lie algebras. In fact in order to obtain the right superalgebra for the supergravity theory $\mathcal{N} = 2$ $d = 4$ (resp. $\mathcal{N} = 2$ $d = 3$), we need to quotient the superBorcherds algebra by the following relation invariant under σ: $\alpha_0 + \beta_2 = \beta_1 + \beta$ (resp. $\alpha_2 + \alpha_1 + \beta_1 = \alpha_0 + \beta_2$). Indeed, the dimension of the Picard group is of the corresponding real del Pezzo is 3 (resp. 4) and therefore, the roots are not independent. This is another instance of a feature that deserves further investigation [33].

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JHEP04(2003)060

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